Lagrange Multipliers 2

This is a follow on sheet to **Lagrange Multipliers 1** and as promised, in this sheet we will look at an example in which the **Lagrange multiplier** λ has a concrete meaning and this will enable us to find the answer to a related optimization problem without having to go through the whole process of solving the Lagrange equations again.

Cobb-Douglas Production Functions

Let q denote the quantity produced of a good. In general this will depend on the amount of capital and labour employed in the production. As a first approximation, we will assume that

$$q = f(K, L),$$

where K denotes capital and L denotes labour.

A **Cobb-Douglas production function** relates the quantities q, K, and L in the following way:

$$q = cK^{\alpha}L^{\beta},$$

where α, β and c are constants and α and β are such that $0 < \alpha < 1$ and $0 < \beta < 1$.

Example

Consider the Cobb-Douglas production function

$$q = 30x^{2/3}y^{3/10},$$

where x represents the number of units of capital and y represents the number of units of labour. Suppose that a firm's unit capital and labour costs are €5 and €6 respectively.

1. Find the values of x and y that maximise output if the total input costs are fixed at €7250.

2. Find the new maximum output if the input costs are increased to €7300.

Solution

1. We have to maximize $q = 30x^{\frac{2}{3}}y^{\frac{3}{10}}$ subject to $5x + 6y = 7250$.

   We let our constraint equation be $g(x) = 5x + 6y - 7250 = 0$.

   Since $\nabla g = (5,6) \neq (0,0)$, there exists a $\lambda \in \mathbb{R}$ such that $\nabla q = \lambda \nabla g$.

   We need to solve $\nabla q = \lambda \nabla g$ and $g = 0$. Now

$$\nabla q = \left(20x^{-\frac{1}{3}}y^{\frac{3}{10}}, 9x^{\frac{2}{3}}y^{-\frac{7}{10}}\right) = \left(\frac{20y^{\frac{3}{10}}}{x^{\frac{1}{3}}}, \frac{9x^{\frac{2}{3}}}{y^{\frac{7}{10}}}\right) = \lambda(5,6).$$

   Thus

$$\frac{20y^{\frac{3}{10}}}{x^{\frac{1}{3}}} = 5\lambda \quad \text{and} \quad \frac{9x^{\frac{2}{3}}}{y^{\frac{7}{10}}} = 6\lambda.$$

   Hence

$$\lambda = \frac{4y^{\frac{3}{10}}}{x^{\frac{1}{3}}} = \frac{3x^{\frac{2}{3}}}{2y^{\frac{7}{10}}} \quad \Longrightarrow \quad 8y = 3x \quad \Longrightarrow \quad y = \frac{3}{8}x.$$

   Substituting this in the constraint equation, we obtain

$$5x + 6 \cdot \frac{3}{8}x = 7250 \quad \Longrightarrow \quad \frac{29}{4}x = 7250 \quad \Longrightarrow \quad x = 1000 \quad \text{and then} \quad y = 375.$$

   We now check the value of q at $(1000, 375)$ and also at the endpoints of the constraint line $5x + 6y = 7250$ to determine where the maximum occurs. Since we are only interested in non-negative values of x and y, the endpoints of the constraint line $5x + 6y = 7250$ lie where where it cuts the x and y axes, that is at $(1450, 0)$ and $\left(0, \frac{3625}{3}\right)$.

   Now

$$q(1450, 0) = q\left(0, \frac{3625}{3}\right) = 0 \quad \text{and} \quad q(1000, 375) \simeq 17755.$$

   Thus the maximum is indeed attained at $x = 1000$, $y = 375$.

2. We will use the fact that if the input costs are increased by €1 then the maximum output is increased by the Lagrange multiplier λ, which in this context is called **The Marginal Productivity of Money**.

   So in this case the marginal productivity of money is

$$\lambda = \frac{4(375)^{\frac{3}{10}}}{(1000)^{\frac{1}{3}}} \simeq 2.37.$$

   Since we are increasing the input costs by €50, the new maximum output is the old maximum output plus 50λ, so it is

$$q(1000, 375) + 50\lambda \simeq 17873.$$